What can we learn from new measurements of Dalitz plot parameters for K $\rightarrow 3\pi$ decays ?

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Abstract

We give a simple expression in linear and quadratic Dalitz–plot slopes which does not depend on the charge combination of the 3π state ($K^\pm \rightarrow \pi^\pm\pi^+\pi^-$ or $\pi^\pm\pi^0\pi^0$ and $K^0_L \rightarrow \pi^+\pi^-\pi^0$ or $\pi^0\pi^0\pi^0$), if all phases between final states are negligible. After investigating the influence of radiative corrections, it is shown how new measurements, especially of quadratic slopes in the $\pi^\pm\pi^0\pi^0$ channel, could help to test theoretical predictions more stringently.

Introduction

At least until the advent of decisive data on CP–violation from the B sector, the K sector remains our main source of experimental information on this old and important question. Further progress depends, however, not only on very high statistics experiments, but also on well tested effective theoretical models for handling long–range strong interactions in a quantitative way.

As in the 3–generation quark model CP–violation is thought to be due to a phase factor in the Cabbibo–Kobayashi–Maskawa–matrix, any measurable first order effect („direct" CP–violation) is expected to disappear if certain phase shifts from (strong) final state interactions (f.s.i.) between different components of transition amplitudes are vanishing. In an attempt to estimate the size of CP–violation effects in K$\rightarrow$ 3π decays [1] it was found necessary not only to fit a number of coefficients determining the weak effective Lagrangian from experimental data, but also to rely on loop calculations in deriving the f.s.i. phases. A further problem is that the strong interaction effects in K$\rightarrow$ 3π decays cannot be handled (like in K$\rightarrow$ 2π) by just multiplying final (strong) eigenstates with constant phase factors; imaginary parts from pion loops depend on dynamical variables, and the real parts are modified in turn by various loop contributions, among them loops with inner K lines. Therefore the denotation as f.s.i. phases is not quite correct. It would be desirable to get more direct experimental information on these phases from existing experimental data on K decays, but this turns out to be very difficult, as we will explain below.

A well documented joint fit of isotopic K$\rightarrow$ $2\pi, 3\pi$ amplitudes has been given by Devlin and Dickey[2], and more recent work is published by Kambor et al.[3]. Both groups of authors found it necessary to make assumptions on the f.s.i. phases in 3π states, namely that they are zero, resp. $< 15^o$. While general unitarity arguments lead us to agree with these expectations concerning the constant parts of the amplitudes, the situation may be more complex for the coefficients (b_{IJ}, c_{IJ}, d_{IJ}, see below) of the non constant parts, which vanish at the center of the Dalitz plot.

Instead of repeating once more these fits, varying merely the data sets used and the parameter sets to be fitted, it may be more useful to investigate by simpler means which data are particularly important for our question, and to check their internal consistency and the appropriateness of the radiative corrections applied before the data can be used to fit isospin amplitudes. Their effect is found to be very important for our investigation, and the standard procedure for handling them (see [4]) may not be sufficient.

In the following sections, we first repeat the definitions and notations (section 1), discuss the internal consistency of slope parameters (section 2), introduce radiative corrections and demonstrate a method to recalculate slope parameters without reference to raw data from older experiments (chapter 3), and finally discuss the significance of future experiments (chapter 4).

Definitions and Notation

In order to fix the notation, we give a short account of relevant definitions for K$\rightarrow 3\pi$ decays connected to isospin relations.

The kinematic (Dalitz plot) variables used are

$$X = \frac{(s_2 - s_1)}{m_{\pi^+}^2}, \quad Y = \frac{(s_3 - s_0)}{m_{\pi^+}^2};$$

$$s_i = (p_K - p_i)^2 \ (i = 1,2,3), \quad s_0 = \frac{1}{3}(s_1 + s_2 + s_3).$$

We further use the three (not independent) variables $r_i = s_i - s_0$ for expansions about the center of the Dalitz plot. We will consider the following transition amplitudes ($K \rightarrow \pi_1\pi_2\pi_3$):

$$\begin{array}{lr}
K^+ \rightarrow \pi^+\pi^+\pi^- : & A^+ = \sum A^+_{IJ}(r_1, r_2, r_3), \\
K^+ \rightarrow \pi^0\pi^0\pi^+ : & A^{+'} = \sum A^{+'}_{IJ}(r_1, r_2, r_3), \\
K^0_L \rightarrow \pi^+\pi^-\pi^0 : & A^0 = \sum A^0_{IJ}(r_1, r_2, r_3), \\
K^0_L \rightarrow \pi^0\pi^0\pi^0 : & A^{0'} = \sum A^{0'}_{IJ}(r_1, r_2, r_3).
\end{array}$$

Factors $(n!)^{-1/2}$ for identical pions are not included. Each A is a sum of isospin contributions with I = final state isospin, contributing to the given channel, and J = two times the isospin change ΔI. The relevant states have $I = 1$ or $I = 2$ for isospin changes $\Delta I = \frac{1}{2}, \frac{3}{2}$. The $(3\pi)_{I=0}$–state is totally antisymmetric and therefore has a totally antisymmetric momentum eigenfunction, that means a form factor $f \sim (r_1 - r_2)(r_1 - r_3)(r_2 - r_3)$, leading to contributions of at least 3^{rd} order in r_i, which we will neglect. $I = 2$ final states are not present in K^0_L decays, if we assume CP invariance ($CP = (-1)^I$ for 3π S-states).

Due to the Wigner-Eckhart-theorem, we have to introduce only 3 independent form factors f_{11}, f_{13}, f_{23}, whereby further restrictions are derived by expanding the amplitudes up to 2^{nd} order and separating different symmetry classes in r_1, r_2, r_3.

Defining $f^{(1)}(r_1, r_2, r_3) = \frac{1}{2}[f(r_1, r_2, r_3) + f(r_1, r_3, r_2)]$, etc., the expansion up to 2^{nd} order in $r_{1,2,3}$ can be written: $f^{(i)}(r_1, r_2, r_3) \approx a + br_i + c(r_1^2 + r_2^2 + r_3^2) + d(2r_i^2 - r_j^2 - r_k^2)$ (other linear and quadratic terms can be included by redefinition of b, c, d, since $r_1 + r_2 + r_3 = 0$). After constructing properly symmetrized $I = 1, 2$ final states from three pion ($I_i = 1$) states (see [5] for details), one finds

$$\begin{aligned}
A^+_{1J} &= f^{(1)}_{1J} + f^{(2)}_{1J}, \\
A^0_{1J} &= g^{(3)}_{1J}, \\
A^{+'}_{1J} &= f^{(3)}_{1J}, \\
A^{0'}_{1J} &= g^{(1)}_{1J} + g^{(2)}_{1J} + g^{(3)}_{1J} \quad (J = 1, 3), \\
A^+_{23} &= A^{+'}_{23} = 2f^{(3)}_{23} - f^{(1)}_{23} - f^{(2)}_{23}
\end{aligned}$$

with

$$-\,g_{1J} = \sqrt{2}\frac{(\frac{J}{2},+\frac{1}{2};\frac{1}{2},-\frac{1}{2}\mid 1,0)}{(\frac{J}{2},+\frac{1}{2};\frac{1}{2},+\frac{1}{2}\mid 1,+1)} f_{1J} \;=\; \{{}^{+1}_{-2}\}\, f_{1J} \tag{1}$$

for $J=1,3$ (the factor $\sqrt{2}$ follows from $K_L^0 = 1/\sqrt{2}(1-CP)K^0$).

Expanding $f_{IJ}^{(i)}$ in the above manner and using

$$r_3 \sim Y\,, \quad r_1^2+r_2^2+r_3^2 \sim Y^2+X^2/3\,, \quad 2r_3^2-r_1^2-r_2^2 \sim Y^2-X^2/3\,, \tag{2}$$

the four amplitudes can be written as

$$A = \sum[a_{IJ}+b_{IJ}Y+c_{IJ}(Y^2+X^2/3)+d_{IJ}(Y^2-X^2/3)] \tag{3}$$

with the coefficients given in table 1.(we write, with slight redefinition, again $\sum a_{IJ} \equiv a$ etc.)

The (irrelevant) relative $(-)$sign between K^0 and K^+ amplitudes has been chosen in accordance with [2]. We do not distinguish at this stage between charge conjugated $K^\pm$ channels and take in the following also averaged experimental data for them.

Due to strong interactions, that means rescattering between initial and/or final states, thef_{IJ} become complex functions of r_i; therefore we have to consider relative phases between all coefficients. Usually a_{11} is chosen to be real and positive.

Consistency of Slope Parameters

From the usual definition of the measurable slope parameters g,h,k (in each channel):

$$\mid A\mid^2 \sim 1+gY+hY^2+kX^2$$

one gets

$$g = 2\,\frac{Re(ab^*)}{\mid a\mid^2}\,, \quad h = \frac{\mid b\mid^2 + 2Re[a(c^*+d^*)]}{\mid a\mid^2}\,, \quad k = \frac{2}{3}\,\frac{Re[a(c^*-d^*)]}{\mid a^2\mid}\,.$$

Since, as mentioned in the introduction, the strong phases for the constant terms a in the isospin amplitudes are thought to be small for general reasons, the real parts of the b - coefficients (after taking a to be real) are determined by the well measured linear slopes g. However, it is easy to see from the isospin relations in tab.1 that neither can the individual contributions b_{IJ} be over constraint (we have 3 measured linear slope parameters g for 3 quantities) nor is it possible to get any information on their imaginary parts or their phases, which are of primary interest here. In order to find these, we must take into account the quadratic slopes h,k. They are much more problematic from the experimental (statistics!) as well as from the theoretical (radiative corrections!) point of

view.
After defining the phases β by

$$b = \mid b \mid e^{i\beta}$$

it is easy to derive for each of the four channels a relation of the form

$$h + 3k - \frac{g^2}{4cos^2\beta} \; = \; 4\,\frac{Re(ac^*)}{\mid a \mid^2} \; \equiv \; R\,. \tag{4}$$

From the isospin components a_{IJ}, c_{IJ} of a, c given in table 1 it is clear that the r.h.s. R of this equation should be the same for the two charged channels and the two neutral channels respectively, independent of any assumptions on the phases of a_{IJ}, c_{IJ}. It should be approximately the same for all four channels if $c_{13} \ll c_{11}$, according to the $\Delta I = \frac{1}{2}$ rule.

Experimental values for g, h, k, and R, calculated from (4) assuming $\beta = 0$, are given in table 2. Instead of using PDG values, we choose to take here in each case the most significant (which in most cases means the latest) experiment. We would have to do this anyway for the $\pi^0\pi^0\pi^+$ and the 3 π^0 data (which have not yet been included completely in the PDG tables). Moreover, because the consistency between some of the experimental data is certainly questionable (compare [2] and [4]), the errors may be more consistent; furthermore it becomes possible to compare our estimate for the effect of the Coulomb corrections with that given in the original paper [6]. In any case, the PDG values do not disagree significantly from those used here.

While the R – values from the two K_L^0 decay experiments are in perfect agreement, the situation for the charged K – decays is less clear, due to the lower (by one order of magnitude) statistics in the $\pi^0\pi^0\pi^+$ channel. If we assume $c_{13} \ll c_{11}$ and take the R value from the (most significant) $\pi^0\pi^0\pi^0$ – experiment [7], we find linear relationships between h and k for all the other channels, as shown in fig.1. The measured (h, k) point should fall above the line, if $\beta \neq 0$. The consistency between the $\pi^\pm\pi^\pm\pi^\mp$ and the $\pi^0\pi^0\pi^0$ data is rather bad; one has to assume a fluctuation of $\sim$ 2 st.dev., or a ratio of $c_{13}\ /\ c_{11} = 0.3 \pm 0.1$, in order to reach agreement. However from a comparison of figs.1a,b which show the results from the same experiment derived without and with Coulomb corrections, respectively, it is clear that this result is very sensitive to these corrections. Therefore we will introduce more complete radiative corrections in the next section.

Radiative Corrections

Radiative corrections for hadronic processes are, generally speaking, model dependent, insofar as structural effects are concerned. The leading contributions can be estimated, however, by calculating the virtual photon exchanges and

soft photon emissions by point–like mesons, describing the weak decay by the most simple local interaction:

$$L_w = c_w\, K\, \pi_1\, \pi_2\, \pi_3\,. \tag{5}$$

To first order in α, the relevant graphs are shown in fig.3a, while structural contributions fig.3b are not considered.

All analytic expressions needed are given below for easy reference. A FORTRAN code for calculating the correction factor as a function of X and Y for all $K \to 3\pi$ channels is available on request (schaale@ifh.de).

The corrected decay probability is given by

$$d\Gamma(s_i) = d\Gamma_0(s_i)\Big\{1 + \frac{\alpha}{2\pi}\Big[\sum_{i=1}^{3} e_0 e_i F_{K\pi_i} - \sum_{i<j} e_i e_j F_{\pi_i\pi_j} + \sum_{i=1}^{3} e_i^2 F_i\Big]\Big\}\,, \tag{6}$$

where $e_i = 0, \pm 1$ are charge factors of K, π_i;

$$\begin{aligned} F_{K\pi_i} = & - \; ln\Big(\frac{2\Delta\varepsilon}{\mu}\Big)^2\Big(2 + \frac{1}{v_i}\, ln\frac{1-v_i}{1+v_i}\Big) + \Big(1 - \frac{\Sigma}{s_i}\Big)\Big[\frac{v_i}{2}\, ln\Big(\frac{1-v_i}{1+v_i}\Big) \\ & + \; 2\int_{-1}^{+1}\frac{dz}{\Phi_i(z)}\,\Big(\,ln\Big|\frac{s_i}{4\mu^2}\Phi_i(z)\Big| - Q_i(z)\, ln\Big|\frac{1-Q_i(z)}{1+Q_i(z)}\Big|\Big)\Big] \\ & - \; \Big(1 - \frac{\Delta}{s_i}\Big)\, ln\frac{m}{\mu} - 6\frac{\mu}{m-\mu}\, ln\frac{m}{\mu} + 8 \end{aligned}$$

and

$$\begin{aligned} F_{\pi_i\pi_j} = & - \; ln\Big(\frac{2\Delta\varepsilon}{\mu}\Big)^2\Big(2 + \frac{1+v_{ij}^2}{v_{ij}}\, ln\frac{1-v_{ij}}{1+v_{ij}}\Big) - \Big(v_{ij}\, ln\frac{1-v_{ij}}{1+v_{ij}} + 2\Big) \\ & + \; (1+v_{ij}^2)\int_{-1}^{+1}\frac{dz}{z^2-v_{ij}^2}\Big(\,ln\Big|\frac{z^2-v_{ij}^2}{1-v_{ij}^2}\Big| - \frac{z^2-v_{ij}^2}{z^2v_{ij}^2-1}Q_{ij}(z)\, ln\Big|\frac{1-Q_{ij}(z)}{1+Q_{ij}(z)}\Big|\Big) - 8\mathbf{G}\,, \end{aligned}$$

where $\mathbf{G} = 0.915966$ is Catalan's constant;

$$F_i = -\frac{1}{v_i}\, ln\frac{1-v_i}{1+v_i}\; - 2\,;$$

$$\Phi_i(z) = z^2 - 2z\frac{\Delta}{s_i} - \Big(1 - \frac{2\Sigma}{s_i}\Big), \quad \Sigma = m^2 + \mu^2\,, \quad \Delta = m^2 - \mu^2\,;$$

$$Q_i(z) = \frac{(1+z)\varepsilon_i + (1-z)m}{(1+z)|\vec{p}_i|}\,, \quad Q_{ij}(z) = \frac{(1+z)\varepsilon_i + (1-z)\varepsilon_j}{|(1+z)\vec{p}_i + (1-z)\vec{p}_j|}\,;$$

$v_i = |\vec{p}_i|/\varepsilon_i$ is the CMS-velocity; $v_{ij} = \sqrt{1 - 4\mu^2/s_k}$ are the velocities in (i,j)-rest system; $m = m_{K^+}$, $\mu = m_{\pi^+}$, and $\Delta\varepsilon$ is the γ cut–off energy. The conventional Coulomb correction factor [8] is given by

$$\prod_{i<j}\eta_{ij}/(exp(\eta_{ij}) - 1)\,, \quad \eta_{ij} = 2\pi\alpha e_i e_j/|\vec{v}_i - \vec{v}_j|\,. \tag{7}$$

In fig.4 are shown the correction factors according to (6), with $\Delta\varepsilon = 10MeV$ are shown as functions of X, Y for all charged channels. They are largest for the $K^{\pm} \to \pi^{\pm}\pi^{+}\pi^{-}$ decay (for this channel we show for comparison also the values calculated with $\Delta\varepsilon = 50MeV$ and with the conventional Coulomb factor (7)). For the other decay modes the corrections are much less important and depend only on Y. For $K^{\pm} \to \pi^{\pm}\pi^{0}\pi^{0}$ the Coulomb factor (7) equals 1, because there is only one charged particle in the final state.

We add some remarks concerning the well known singularities appearing in the treatment of radiative corrections, in order to show the limits of this approach.

The first kind are ultraviolet divergences of the loop integrals which result in explicit cut–off dependent terms $\sim\ ln(\Lambda/\mu)$. The reason is that we have chosen purely local effective interactions without hadronic form factors, which would regularize these integrals. In our approach we renormalized them by the requirement that their contributions should disappear for transitions, in which the incoming charge reemerges as outgoing with the same velocity. This means we subtract $F_{K\pi}(s_i = (m-\mu)^2)$ and $F_{\pi\pi}(s_i = 0)$ respectively (standard renormalization on mass shell)[1]. As we neglected higher order corrections to L_w ($L = L_w + \eta L_1$, with $\eta \sim 0.1$, say), being responsible for the kinematic structure of the Dalitz–plot, our results for slope parameters are correct up to terms of order $\alpha\eta$. The constant terms a_{IJ} are affected by the unknown renormalization ambiguity of order α; in other words, hadron structure effects may differ between $K_L^0 \to 3\pi^0$ and charged K decays by terms of order α. It is easy to show that this induces also corrections of order $\alpha\eta$ for slope parameters. Their treatment would require a more complicated effective Lagrangian, inclusion of structural photons etc., and would be strongly model–dependent.

A second kind are so–called collinear singularities, appearing in the case of at least two charged particles with equal velocities in the final state, that means on the Dalitz–plot boundary. In these points the perturbation expansion breaks down. Accepting that some regions of the Dalitz-plot are just not handled correctly by the theory, some caution in the treatment of experimental data near these singularities may be required. It can be shown that no infinities are encountered in integrals over kinematic regions (integrability).

The third and last type are infrared divergences caused by low energy photons. Their cure is well known, leading to the introduction of the upper limit $\Delta\varepsilon$, defined here in the K rest system, up to which weak photons are to be included in the definition of the decay channel. A rough estimate of $\Delta\varepsilon$ may be derived from the mass resolution achieved in reconstructing the K mass. However, care should be exercised when high energy K–decays are analyzed. A correct experimental treatment would have to include radiative processes in

[1] The subtraction constants include also all contributions from photon emission graphs; in this respect the present results differ from those given in [12]

the Monte–Carlo and to establish an effective $\Delta\varepsilon$ in this way. [2]

In order to estimate the influence of the radiative and the Coulomb corrections on the Dalitz–plot parameters without reference to the experimental data, i.e. uncorrected Dalitz–plot densities, we introduce moments $< X^m \cdot Y^n >^{(\prime)}$ with respect to normalized Dalitz–plot densities $p(X,Y)$ and $p'(X,Y)$, where p represents a constant density and p' includes radiative corrections to this constant density. After expanding p' in X, Y, it is easy to get approximate expressions for p' moments $< ... >'$ in terms of p moments $< ... >$, but for our case we have calculated all relevant moments numerically. If we suppose that the experimental data sample is already corrected for experimental efficiency and background [3]. , the slope parameters are to be derived from a fit to the Dalitz–plot density, which may be written as

$$f(X,Y) = \frac{1 + \vec{v} \cdot \vec{g}}{1 + < \vec{v} > \cdot \vec{g}}\, p(X,Y)\,, \tag{8}$$

where for convenience the vectors

$$\vec{v} = \begin{pmatrix} Y \\ Y^2 \\ X^2 \end{pmatrix} \qquad\qquad \vec{g} = \begin{pmatrix} g \\ h \\ k \end{pmatrix}$$

are introduced. We write $\vec{v}(i) = \vec{v}(X_i, Y_i)$ for measured X_i and Y_i for the ith event. $\vec{g}$ is to be estimated by the Maximum Likelihood method from the Likelihood function

$$L = \prod_{i=1}^{n} f(X_i, Y_i) \tag{9}$$

leading to the system of equations for $\vec{g}$

$$\frac{1}{n} \sum_i \frac{\vec{v}(i)}{1 + \vec{v}(i) \cdot \vec{g}} = \frac{< \vec{v} >}{1 + < \vec{v} > \cdot \vec{g}} \tag{10}$$

(we do not distinguish here between estimates and population values). If we now apply a further correction, e.g. $p \to p'$, we get, with the same sample of experimental data X_i, Y_i, corrected parameters g', h', k', which can be expressed in terms of $\vec{g}$ and moments $< ... >, < ... >'$. The equations for $\vec{g}'$ are:

$$\frac{1}{n} \sum_i \frac{\vec{v}(i)}{1 + \vec{v}(i) \cdot \vec{g}'} = \frac{< \vec{v} >'}{1 + < \vec{v} >' \cdot \vec{g}'}\,. \tag{11}$$

[2] There are significant differences, at least concerning the parameter k, between [6] and a later experiment [9], which found $k = -0.0205 \pm 0.0039$ to be compared with -0.0075 ± 0.0019 [6] for $K^+ \to \pi^+\pi^+\pi^-$. As the first experiment measured only the momentum of the odd pion in the final state, the second one all momenta, effects of the above mentioned kind may be present. [9] do not present data without (Coulomb) correction, therefore we used only [6].

[3] In the actual evaluation of an experiment, p would be taken from a Monte–Carlo simulation of the measured Dalitz–plot distribution, using a constant Dalitz–plot density as input.

For small corrections we may expand both sides in $\Delta\vec{g} = \vec{g}' - \vec{g}$ and obtain a linearized set of equations:

$$V \cdot \Delta\vec{g} = \frac{<\vec{v}>}{1+<\vec{v}>\cdot\vec{g}} - \frac{<\vec{v}>'}{1+<\vec{v}>'\cdot\vec{g}}\,, \tag{12}$$

where the symmetric matrix V has the elements

$$V_{kl} = \frac{1}{n}\sum_i \Big(\frac{v_k v_l}{(1+\vec{v}\cdot\vec{g})^2}\Big)_{\vec{v}=\vec{v}(i)} - \frac{<v_k>'<v_l>'}{(1+<\vec{v}>'\cdot\vec{g})^2} \quad (k,l=1,2,3)\,. \tag{13}$$

The dependence on X_i, Y_i can now be eliminated by replacing $\frac{1}{n}\sum$ with $\int dXdY\,f(X,Y)$, leading to

$$V_{kl} = \frac{1}{1+<\vec{v}>\cdot\vec{g}}\Big\langle\frac{v_k v_l}{1+\vec{v}\cdot\vec{g}}\Big\rangle - \frac{<v_k>'<v_l>'}{(1+<\vec{v}>'\cdot\vec{g})^2}\,, \tag{14}$$

where the first terms on the right–hand side are also to be evaluated for given g, h, k numerically.

Surely one has to be aware of the severe limitations of the above approach if, for experimental or other reasons, the uncorrected parameters $\vec{g}$ do not represent the density over the whole Dalitz–plot. To give an extreme example, suppose they had been derived from a fit to the Dalitz–plot density in a region where the corrections disappear. They were then found identical to the corrected parameters, if the "corrections" are applied to the raw data sample. For our method one has to assume however that the uncorrected parameters fit the density equally well over the whole Dalitz–plot. If, as we may further suppose in our example, there are sizeable corrections to the density in the unmeasured region this is not the case, and consequently we find the corrected parameters different from the uncorrected ones, possibly even outside the statistical errors.

As a check of our method we calculated for the conventional Coulomb correction factor the corrected parameters $\vec{g}'$, corresponding to the first column of table 2, from the uncorrected values of ref.[6] in the second column. The results are given in table 3 together with the differences with respect to the corrected values from ref.[6].

Our conclusion from this comparison is that the method is useful to demonstrate the influence of radiative corrections on the quadratic slope parameters, where statistical errors are relatively large. It is not a substitute, however, for a complete (re)analysis of precision experimental data like those existing for linear slope parameters, for which systematic corrections are more subtle.

Results and Conclusions

The results for $\Delta g, \Delta h, \Delta k$ are shown as a function of the linear slope g in fig.5 and for the actual g–values (see table 2) in table 4. The dependence on h and k is for small values of these parameters negligible, they are set to zero everywhere.

For the $K^\pm \to \pi^\pm\pi^+\pi^-$ channel the discrepancy of the R–value with that for $K_L^0 \to 3\pi^0$ (see table 2) disappears after applying the radiative corrections (6) instead of the Coulomb–factor (7)(compare fig.1a) with fig.2). We find

$$R^{\pm+-}_{rad.corr.}(\beta = 0) = -.0078 \pm .0089,$$

indicating that the quadratic coefficients c in the decay amplitude (3) may be small and of the same order as for K_L^0–decays.

The corresponding corrections for the channel $K^\pm \to \pi^\pm\pi^0\pi^0$ are smaller than the numerical accuracies of the numbers in table 2. A comparison of R–values of the different charged K–decay channels with comparable statistics would be interesting.

In order to derive some first information on possible phases β we may use the relation

$$R(\beta) = R(0) - \frac{g^2}{4}\tan^2\beta\,.$$

For the $K_L^0 \to \pi^+\pi^-\pi^0$–channel we can identify $R^{+-0}(\beta)$ with R^{000} and find from the last row of tab. 2[4]

$$\tan^2\beta = 0.0 \pm .086\,,$$

i.e. a phase angle $\beta^{+-0} \leq 16^o$. For the charged kaon decay $K^\pm \to \pi^\pm\pi^+\pi^-$, assuming $c_{13} \ll c_{11}$ and comparing R^{000} also with $R^{\pm+-}_{rad.corr.}$, we find analogously $\beta^{\pm+-} \leq 43^o$.

Despite comparable errors of the R–values for the two cases, the restriction on $\beta^{\pm+-}$ is weaker than that found on β^{+-0}. This is due to the different linear slopes g. In view of this, and also because it is much less influenced by radiative corrections, the channel $K^\pm \to \pi^\pm\pi^0\pi^0$ (with $g \sim 0.6$) deserves special attention by experiment. Enhancing the data sample for this channel by an order of magnitude (to $\sim 5\cdot 10^5$ events) could lead to a determination of the phase β with an error $\leq 15^o$ similar as for K_L^0 decays. Clearly, this would help a lot to constrain effective Lagrangian models with regard to higher order (p^4–, loop–, penguin–) contributions, especially if taken together with results from radiative K–decays, where also new experimental and theoretical work is going on.

Besides this, one should be aware that also for the other channels considered here one has to rely presently on only few large statistics experiments. Concerning the presentation of new data, we would like to advocate to publish the data also in a form uncorrected for radiative or Coulomb effects. Furthermore, one should clearly state the regions of the Dalitz–plot to which the slope parameters have been fitted. There is room for later improvements of the radiative corrections, including structural radiation, after more realistic effective Lagrangians will have been introduced.

[4]radiative corrections by the authors [10] are already included

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Channel	$++-$	$00+$	$+-0$	000
a	$2(a_{11}+a_{13})$	$a_{11}+a_{13}$	$-(a_{11}-2a_{13})$	$-3(a_{11}-2a_{13})$
b	$-(b_{11}+b_{13})+b_{23}$	$b_{11}+b_{13}+b_{23}$	$-(b_{11}-2b_{13})$	0
c	$2(c_{11}+c_{13})$	$c_{11}+c_{13}$	$-(c_{11}-2c_{13})$	$-3(c_{11}-2c_{13})$
d	$-(d_{11}+d_{13})+d_{23}$	$d_{11}+d_{13}+d_{23}$	$-(d_{11}-2d_{13})$	0

Table 1: Isospin Amplitudes

Ch.	$\pi^{\pm}\pi^{\pm}\pi^{\mp}$		$\pi^0\pi^0\pi^+$	$\pi^+\pi^-\pi^0$	$\pi^0\pi^0\pi^0$
Expt.	[6],a	[6],b	[11]	[10]	[7]
g	$-.2173\pm.0026$	$-.1866\pm.0025$	$.575\pm.022$	$.677\pm.010$	0
h	$.0156\pm.0062$	$.00125\pm.0062$	$.021\pm.023$	$.079\pm.007$	$-.0033\pm.0013$
k	$-.0079\pm.0019$	$.0029\pm.0021$	$.011\pm.007$	$.0097\pm.0018$	$h/3$
R	$-.0199\pm.0084$	$.0013\pm.0088$	$-.029\pm.032$	$-.0065\pm.0095$	$-.0066\pm.0026$

Table 2: Experimental Data (a with, b without Coulomb correction)

	calc. by (12)	Diff. to [6]
g	$-.2236\pm.0025$	$.0063\pm.0037$
h	$.0149\pm.0062$	$.0007\pm.0088$
k	$-.0079\pm.0021$	$0.0\pm.003$

Table 3: Comparison for Coulomb corrections

Channel	$\pm+-$	±00	$+-0$
Δg	$-1.99\cdot10^{-2}$	$-1.45\cdot10^{-3}$	$7.02\cdot10^{-3}$
Δh	$7.28\cdot10^{-3}$	$-8.47\cdot10^{-4}$	$1.63\cdot10^{-3}$
Δk	$-4.72\cdot10^{-3}$	$7.66\cdot10^{-7}$	$-1.70\cdot10^{-5}$

Table 4: Radiative Corrections for Slope Parameters

Figure captions

Fig. 1 Plots of quadratic slope parameters k vs. h from table 2. The linear relation (4) is indicated with 1 s.d. errors.

$+ + -$ a	: channel $K^{\pm} \to \pi^{\pm}\pi^{+}\pi^{-}$ with Coulomb corrections (7)
b	: uncorrected
$+ \; 0 \; 0$	: channel $K^{\pm} \to \pi^{\pm}\pi^{0}\pi^{0}$, uncorrected
$+ \; - \; 0$	: channel $K_L^0 \to \pi^{+}\pi^{-}\pi^{0}$, corrected by [10]

Fig. 2 The same as fig.1 for channel $K^{\pm} \to \pi^{\pm}\pi^{+}\pi^{-}$ corrected according to (12) with rad. corr. (6)

Fig. 3 a) Graphs for radiative corrections to first order
b) Graphs with inclusion of structural radiation

Fig. 4 Plots of correction factors as functions of X, Y
a) Rad. corr. for $K^{\pm} \to \pi^{\pm}\pi^{+}\pi^{-}$, $\Delta\varepsilon = 10MeV, 50MeV$ (broken lines)
b) Coul. corr. for $K^{\pm} \to \pi^{\pm}\pi^{+}\pi^{-}$
c) Rad. corr. for $K_L^0 \to \pi^{+}\pi^{-}\pi^{0}$
d) Rad. corr. for $K^{\pm} \to \pi^{\pm}\pi^{0}\pi^{0}$

Fig. 5 a) - c) Dependence of $\Delta g, \Delta h, \Delta k$ on g (for $h = k = 0$) for the channel $K^{\pm} \to \pi^{\pm}\pi^{+}\pi^{-}$